

Problemas – Tema 6

Solución a problemas de repaso y ampliación del Tema 5 y Tema 6 - Hoja 06 - Problemas 1, 2, 4

Hoja 6. Problema 1

Resuelto por Isabel Navarro-Pelayo Torres (marzo 2015)

1. Resolver

a) $\int \frac{1}{\cos^4(x)} dx$

b) $\int \frac{1}{\cos^2(x) \cdot \operatorname{sen}^2(x)} dx$

c) $\int \frac{1}{2+\cos(x)} dx$

a) $\int \frac{1}{\cos^4(x)} dx$

Integral par en el producto seno · coseno → cambio de variable $\operatorname{tg}(x) = t$

$$\operatorname{tg}(x) = t \rightarrow dx = \frac{dt}{1+t^2}$$

$$\cos(x) = \frac{1}{\sqrt{1+t^2}}$$

Sustituimos en la integral de partida

$$I = \int \frac{1}{\left(\frac{1}{\sqrt{1+t^2}}\right)^4} \cdot \frac{dt}{1+t^2} = \int \frac{(1+t^2)^2}{1+t^2} dt = \int (1+t) dt = t + \frac{t^2}{2} + C$$

Deshacemos el cambio: $I = \operatorname{tg}(x) + \frac{\operatorname{tg}^2(x)}{2} + C$

$$b) \int \frac{1}{\cos^2(x) \cdot \operatorname{sen}^2(x)} dx$$

Par en el producto seno · coseno → cambio de variable $\operatorname{tg}(x) = t \rightarrow dx = \frac{dt}{1+t^2}$

$$\cos(x) = \frac{1}{\sqrt{1+t^2}}, \quad \operatorname{sen}(x) = \frac{t}{\sqrt{1+t^2}}$$

Sustituimos en la integral de partida.

$$I = \int \frac{1}{\left(\frac{1}{\sqrt{1+t^2}}\right)^2 \cdot \left(\frac{t}{\sqrt{1+t^2}}\right)^2} \cdot \frac{dt}{1+t^2} = \int \frac{(1+t^2) \cdot (1+t^2)}{t^2} \cdot \frac{dt}{1+t^2} = \int \frac{1+t^2}{t^2} dt$$

$$I = \int \frac{1}{t^2} dt + \int dt = \frac{-1}{t} + t + C$$

Deshacemos el cambio: $I = \frac{-1}{\operatorname{tg}(x)} + \operatorname{tg}(x) + C$

$$c) \int \frac{1}{2+\cos(x)} dx$$

Cambio de variable: $\operatorname{tg}(x/2) = t \rightarrow dx = \frac{2}{1+t^2} dt$; $\cos(x) = \frac{1-t^2}{1+t^2}$

$$I = \int \frac{1}{2 + \frac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2} dt = \int \frac{2}{2(1+t^2) + 1 - t^2} dt = 2 \int \frac{1}{t^2 + 3} dt = 2 \int \frac{1}{3 \left(1 + \frac{t^2}{3}\right)} dt = \frac{2}{3} \int \frac{1}{1 + \left(\frac{t}{\sqrt{3}}\right)^2} dt$$

$$I = \frac{2\sqrt{3}}{3} \int \frac{1/\sqrt{3}}{1 + \left(\frac{t}{\sqrt{3}}\right)^2} dt = \frac{2\sqrt{3}}{3} \operatorname{arctg}\left(\frac{t}{\sqrt{3}}\right) + C$$

Deshacemos el cambio: $I = \frac{2\sqrt{3}}{3} \operatorname{arctg}\left(\frac{\operatorname{tg}\left(\frac{x}{2}\right)}{\sqrt{3}}\right) + C$

Hoja 6. Problema 2

Resuelto por Carmen Martín Rubio (marzo 2015)

2. Resolver.

a) $\int \frac{1+e^x}{1-e^x} dx$

b) $\int x^2 \cdot \ln(x) dx$

c) $\int x^2 \cdot \text{sen}(3x) dx$

a) $\int \frac{1+e^x}{1-e^x} dx$

Cambio de variable $e^x = t \rightarrow e^x dx = dt \rightarrow dx = \frac{dt}{t}$

$$\int \frac{1+t}{1-t} \frac{dt}{t} = \int \frac{1+t}{t(1-t)} dt$$

Método de coeficientes indeterminados

$$\frac{1+t}{t(1-t)} = \frac{A}{t} + \frac{B}{1-t}$$

$$1+t = A(1-t) + Bt$$

$$\text{Si } t = 1 \rightarrow 2 = 0 + B \rightarrow B = 2$$

$$\text{Si } t = 0 \rightarrow 1 = A + 0 \rightarrow A = 1$$

$$\int \frac{1+t}{t(1-t)} dt = \int \frac{1}{t} + \frac{2}{1-t} dt = \int \frac{1}{t} dt + 2 \int \frac{1}{1-t} dt$$

$$\ln(t) - 2 \ln(1-t) + C$$

Deshago cambio de variable

$$I = \ln(e^x) - 2 \ln(e^x - 1) + C = x - 2 \ln(e^x - 1) + C$$

b) $\int x^2 \cdot \ln(x) dx$

Integro por partes

$$u = \ln(x) \rightarrow du = \frac{1}{x} dx$$

$$dv = x^2 dx \rightarrow v = \frac{x^3}{3}$$

$$\int u dv = u \cdot v - \int v \cdot du$$

$$\int x^2 \cdot \ln(x) dx = \ln(x) \cdot \frac{x^3}{3} - \int \frac{x^3}{3} \cdot \frac{1}{x} dx$$

$$\int x^2 \cdot \ln(x) dx = \ln(x) \cdot \frac{x^3}{3} - \int \frac{x^3}{3x} dx$$

$$\int x^2 \cdot \ln(x) dx = \ln(x) \cdot \frac{x^3}{3} - \frac{1}{3} \int x^2 dx$$

$$\int x^2 \cdot \ln(x) dx = \ln(x) \cdot \frac{x^3}{3} - \frac{x^3}{9}$$

$$\int x^2 \cdot \ln(x) dx = \frac{1}{3} x^3 \left(\ln(x) - \frac{1}{3} \right) + C$$

c) $\int x^2 \cdot \text{sen}(3x) dx$

Integramos por partes

$$u = x^2 \rightarrow du = 2x dx$$

$$dv = \text{sen}(3x) dx \rightarrow v = \frac{-1}{3} \cos(3x)$$

$$\int u dv = u \cdot v - \int v \cdot du$$

$$\int x^2 \cdot \text{sen}(3x) dx = x^2 \cdot \frac{-1}{3} \cdot \cos(3x) - \int \frac{-1}{3} \cdot \cos(3x) \cdot 2x dx$$

$$\int x^2 \cdot \text{sen}(3x) dx = x^2 \cdot \frac{-1}{3} \cdot \cos(3x) + \frac{2}{3} \int \cos(3x) \cdot x dx$$

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Realizamos la integral por partes I'

$$\begin{aligned} u = x &\rightarrow du = dx \\ dv = \cos(3x) dx &\rightarrow v = \frac{1}{3} \cdot \operatorname{sen}(3x) \end{aligned}$$

$$\int u dv = u \cdot v - \int v \cdot du$$

$$\int \cos(3x) \cdot x dx = x \cdot \frac{1}{3} \cdot \operatorname{sen}(3x) - \int \frac{1}{3} \cdot \operatorname{sen}(3x) \cdot dx$$

$$\int \cos(3x) \cdot x dx = x \cdot \frac{1}{3} \cdot \operatorname{sen}(3x) - \frac{1}{3} \int \operatorname{sen}(3x) \cdot dx$$

$$\int \cos(3x) \cdot x dx = x \cdot \frac{1}{3} \cdot \operatorname{sen}(3x) - \frac{1}{3} \cdot \left(\frac{-1}{3}\right) \int (-3) \cdot \operatorname{sen}(3x) \cdot dx$$

$$\int \cos(3x) \cdot x dx = x \cdot \frac{1}{3} \cdot \operatorname{sen}(3x) + \frac{1}{9} \cdot \cos(3x) + C$$

Por tanto sustituyo este valor en la integral de partida:

$$I = x^2 \cdot \frac{-1}{3} \cdot \cos(3x) + \frac{2}{3} \left(x \cdot \frac{1}{3} \cdot \operatorname{sen}(3x) + \frac{1}{9} \cdot \cos(3x) \right) + C$$

$$I = \frac{-x^2 \cdot \cos(3x)}{3} + \frac{2x \cdot \operatorname{sen}(3x)}{9} + \frac{\cos(3x)}{27} + C$$

$$I = \frac{-9x^2 \cdot \cos(3x) + 6x \cdot 3 \operatorname{sen}(3x) + \cos(3x)}{27} + C$$

Hoja 6. Problema 4

Resuelto por Carmen Jiménez (marzo 2015)

1. Resolver

a) $\int \frac{3^x}{1+3^x} dx$

b) $\int \frac{1}{\sqrt{4-x^2}} dx$

c) $\int \sqrt{\frac{x+2}{x-1}} dx$

a) $\int \frac{3^x}{1+3^x} dx = \frac{1}{\ln(3)} \int \frac{\ln(3) \cdot 3^x}{1+3^x} dx = \frac{1}{\ln(3)} \cdot \ln|1+3^x| + C$

b) $\int \frac{1}{\sqrt{4-x^2}} dx$

Proponemos el cambio de variable $x = 2\text{sen}(t) \rightarrow dx = 2\cos(t) dt$

$$I = \int \frac{1}{\sqrt{4-4\text{sen}^2(t)}} 2\cos(t) dt = \int dt = t + C$$

Deshacemos el cambio de variable.

$$I = \text{arccosen}\left(\frac{x}{2}\right) + C$$

c) $\int \sqrt{\frac{x+2}{x-1}} dx$

Aplicamos el cambio de variable $\frac{x+2}{x-1} = t^2 \rightarrow \frac{x-1-(x+2)}{(x-1)^2} dx = 2t dt \rightarrow \frac{-3}{(x-1)^2} dx = 2t dt \rightarrow$

$$\rightarrow dx = \frac{-2t}{3}(x-1)^2 dt$$

$$I = \int \sqrt{\frac{x+2}{x-1}} dx = \int t \cdot \frac{-2t}{3}(x-1)^2 dt$$

Debemos expresar x en función de $t \rightarrow \frac{x+2}{x-1} = t^2 \rightarrow x+2 = x \cdot t^2 - t^2 \rightarrow 2+t^2 = x \cdot t^2 - x \rightarrow$

$$\rightarrow 2+t^2 = x(t^2-1) \rightarrow x = \frac{2+t^2}{t^2-1}$$

Sustituimos

$$I = \int \frac{-2t^2}{3} \left(\frac{2+t^2}{t^2-1} - 1 \right)^2 dt = \int \frac{-2t^2}{3} \left(\frac{3}{t^2-1} \right)^2 dt = -6 \int \frac{t^2}{(t^2-1)^2} dt = -6 \int \frac{t^2}{(t+1)^2 \cdot (t-1)^2} dt$$

Aplicamos el método de los coeficientes indeterminados:

$$\frac{t^2}{(t+1)^2 \cdot (t-1)^2} = \frac{A}{t+1} + \frac{B}{(t+1)^2} + \frac{C}{t-1} + \frac{D}{(t-1)^2}$$

$$I = -6 \left(A \ln|t+1| - B \frac{1}{t+1} + C \ln|t-1| - D \frac{1}{t-1} \right)$$

Dando valores, obtenemos:

$$A = \frac{-1}{4}$$

$$B = \frac{1}{4}$$

$$C = \frac{1}{4}$$

$$D = \frac{1}{4}$$

Por lo que la integral queda:

$$I = -6 \left(\frac{-1}{4} \ln|t+1| - \frac{1}{4} \frac{1}{t+1} + \frac{1}{4} \ln|t-1| - \frac{1}{4} \frac{1}{t-1} \right)$$

Deshacemos el cambio de variable $\frac{x+2}{x-1} = t^2 \rightarrow \sqrt{\frac{x+2}{x-1}} = t$

$$I = \frac{3}{2} \ln \left| \sqrt{\frac{x+2}{x-1}} + 1 \right| + \frac{3}{2} \frac{1}{\sqrt{\frac{x+2}{x-1}} + 1} - \frac{3}{2} \ln \left| \sqrt{\frac{x+2}{x-1}} - 1 \right| + \frac{3}{2} \frac{1}{\sqrt{\frac{x+2}{x-1}} - 1} + C$$